Koszul duality + equivalences of categories – Matthew

<u>Recall/reboot</u>: $A = \bigoplus_{i \geq 0} A_i$ graded ring/algebra

- $A_0 = k$ semi-simple (often a field)
- A_i fin. gen over k

$$A^! := \mathrm{Ext}^\bullet_A(k,k)^{op} \qquad \text{Note: } A^!_0 = k$$

<u>Def'n</u> A is <u>Koszul</u> if $\exists$ (graded) projective resolution

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow k \longrightarrow 0$$

s.t. P^i is generated in deg i.

<u>Rem</u>: $\Leftrightarrow (A^!)^! \xrightarrow{\sim} A$

$\Leftrightarrow$ A quadratic with quadratic dual $(A^!)^{op}$

$\Rightarrow$ $A^!$ is Koszul.

One resolution is $A \otimes_k (A^!_\bullet)^{\vee} \longrightarrow k \longrightarrow 0$, where $(A^!_i)^\vee = \mathrm{Hom}_k(A^!_i, k)$

..."Koszul complex",

if A is Koszul, "Koszul resolution"

<u>Goal</u>: Relate representation theory of $A, A^!$.

1) Construct adjunction $K(A) \underset{G}{\overset{F}{\rightleftarrows}} K(A^!)$ between

"<u>homotopy categories</u> of chain complexes"

<u>objects</u>: chain complexes of gr. modules

<u>morphisms</u>: chain maps modulo homotopy

2) See that this does <u>not</u> induce an equivalence $D(A) \rightleftarrows D(A^!)$

of "derived categories"

<u>morphisms</u>:

3) What can we do instead?

1) <u>Idea:</u> In $D(A^!)$, $A^! = R\mathrm{Hom}_A(k,k)^{op}$

Compute this by using Koszul complex.

<u>Fix data:</u> · quadratic presentation of A: $A \xleftarrow{\sim} T(V)/R$,

V fin. dim'l v.sp. (k field)

· $\mathrm{Hom}_k(V,V) \xrightarrow{\sim} \mathrm{Hom}_k(V,k) \otimes_k V$

$1 \longmapsto \sum v_\alpha^\vee \otimes v_\alpha$

i.e. basis + dual basis of V

$F: K(A) \longrightarrow K(A^!)$

$FM = A^! \otimes_k M = \bigoplus_{\ell,i} A^!_\ell \otimes M^i$ as bigraded v.sp.

$\left(= \bigoplus \mathrm{Hom}_A(A \otimes (A^!_\ell)^*, M^i)\right)$ cf. $R\mathrm{Hom}_A(k,-)$ (M) using preferred Koszul resol.

Degree notation: $X^{\cdot}_{\cdot}$ ← homol. degree, diff increases; ← grading

$$(FM)^p_q = \bigoplus_{\substack{p=i+j \\ q=\ell-j}} A^!_\ell \otimes M^i_j \qquad d_{FM}(a \otimes m) = (-1)^{i+j} \sum a v_\alpha^\vee \otimes v_\alpha m + d \otimes d_M(m)$$

$a \in A^!_\ell$, $m \in M^i_j$

$G: K(A^!) \longrightarrow K(A)$

$GN = \mathrm{Hom}_k(A,N) = \bigoplus_{\ell,i} \mathrm{Hom}_k(A_{-\ell}, N^i)$

$\left(= \bigoplus A^*_{-\ell} \otimes A^! \otimes_{A^!} N^i\right)$ cf. derived tensor product using preferred Koszul resol.

$$(GN)^p_q = \bigoplus_{\substack{p=i+j \\ q=\ell-j}} \mathrm{Hom}_k(A_{-\ell}, N^i_j)$$

$$d_{GN} f = (-1)^i \sum v_\alpha^\vee f \circ (v_\alpha \cdot -) + d_N \circ f$$

<u>Thm</u> (Beilinson – Ginzburg – Soergel)

$$K(A) \underset{G}{\overset{F}{\rightleftarrows}} K(A^!) \quad \text{is an adjunction}$$

(+ Fløystad) Similar for $A = \mathcal{U}(\mathfrak{g})$

2)
<u>Example</u>: $A = k[t]$, k a field
$= T(k) = \mathrm{Sym}(k) = \mathcal{U}(\mathbb{G}_a)$

$$\Rightarrow A^! = k \oplus k\varepsilon \cong k[\varepsilon]/(\varepsilon^2)$$

simple (ungraded) A-modules are $k[t]/(t-a) =: S_a$, $k = S_0$

Koszul resolution: $0 \to k[t] \xrightarrow{t} k[t] \to k \to 0$

Compute $F(S_a)$:

$$\begin{array}{ccc} \mathrm{Hom}_{k[t]}(k[t], S_a) & \xrightarrow{t^*} & \mathrm{Hom}_{k[t]}(k[t], S_a) \\ \cong & & \cong \\ S_a & \xrightarrow[\cdot a]{} & S_a \end{array}$$

is isomorphism for $a \neq 0$!

So, in $D(A^!)$, $F(S_a) = \begin{cases} 0 & a \neq 0 \\ A^! & a = 0 \end{cases}$

$\Rightarrow$ <u>need</u> to restrict to <u>graded</u> A-modules.

Indecomposable Graded A-modules are $k[t]/(t^n)$ and $k[t]$

$$F(k[t]/(t^n)): \quad k[t]/(t^n) \xrightarrow{\cdot t} k[t]/(t^n)$$

$$F(k[t]): \quad k[t] \xrightarrow{\cdot t} k[t]$$

- is <u>not</u> 0 :)
- are <u>different</u> in $D(A^!)$

$\simeq$ $k[-1]\langle 1 \rangle$

quasi-iso. homol. degree grading

Remarks:

. This is a shadow of BGG correspondence

$$\begin{array}{ccc} \mathbb{D}^b(\mathrm{grmod}\ \mathrm{Sym}\, V) & \xrightarrow{\sim} & \mathbb{D}^b(\mathrm{grmod}\ \Lambda V^*) \\ \wr\wr & & \\ \mathbb{D}^b(\mathrm{Coh}\ \mathbb{P}V) & & \end{array}$$

. Don't get equivalence: $D(k[t]) \longrightarrow D(k[\varepsilon]/(\varepsilon^2))$

b/c $k[t]$ is compact, but $F(k[t])$ is not compact.

3) Can we do better?

Problem: F, G don't preserve acyclicity of complexes

1st solution (BGS) There is an induced equivalence

$$D^{\downarrow}(A) \underset{\longleftarrow}{\xrightarrow{\sim}} D^{\uparrow}(A^!)$$

between full subcategories defined by boundedness conditions

$\Downarrow$

can use spectral sequences

(Corollary: BGG)

3 a) Fløystad. abstract nonsense: $\mathcal{A}, \mathcal{B}$ triangulated categories, and

adjunction $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$

Def'n: $\mathcal{C}$ triang. cat, A full subcategory $\mathcal{Z} \subseteq \mathcal{C}$ is a null-system

if i) $\mathcal{Z}$ is closed under isom, Σ and Σ^{-1}, and $0 \in \mathcal{Z}$

ii) if $L \to M \to N \to \Sigma L$ is a triangle in $\mathcal{C}$

st $L, N \in \mathcal{Z} \Rightarrow M \in \mathcal{Z}$.

If $\mathcal{Z}$ is a null-system, then

$W(\mathcal{Z}) = \{ f : X \to Y : \mathrm{Cone}(f) \in \mathcal{Z} \}$ satisfies 2 out of 3 prop (is multiplicative).

$\rightsquigarrow$ $W(\mathcal{Z})$ is a notion of weak equivalence wrt which we can localize!

$$\mathcal{C}[W(Z)^{-1}] =: \mathcal{C}/Z$$

<u>Example</u>: $\mathcal{C} = K(A)$, Z = acyclic complexes

$\Rightarrow W(Z)$ = quasi-isos, $\mathcal{C}/Z = D(A)$.

<u>Def'n</u>
Pick null systems $Z_{\mathcal{A}}, Z_{\mathcal{B}}$ on $\mathcal{A}, \mathcal{B}$. They are <u>compatible with adjunction</u> if units $a \longrightarrow GF(a)$ & counits $FG(b) \longrightarrow b$ lie in $W(Z_{\mathcal{A}}), W(Z_{\mathcal{B}})$

$\Longrightarrow$ get new null systems

$$N_{\mathcal{A}} = \{a \in Z_{\mathcal{A}} : F(a) \in Z_{\mathcal{B}}\}$$

$$N_{\mathcal{B}} = \{b \in Z_{\mathcal{B}} : G(b) \in Z_{\mathcal{A}}\}$$

<u>Thm</u> [Fløystad]
F, G induce equivalences

$$\mathcal{A}/N_{\mathcal{A}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}/N_{\mathcal{B}}$$

Apply to $K(A) \underset{G}{\overset{F}{\rightleftarrows}} K(A^!)$ with $Z_{\mathcal{A}}, Z_{\mathcal{A}^!}$ given by acyclics.
i.e. forced that we localized wrt those that send acyclics to acyclics.

$$\begin{tikzcd}
K(A) \arrow[r, shift left, "F"] \arrow[r, phantom, "\simeq"] \arrow[d, "\text{localize}"'] & K(A^!) \arrow[l, shift left, "G"] \arrow[d] \\
\mathcal{C}_A \arrow[r, shift left, "F"] \arrow[r, phantom, "\simeq"] \arrow[d, "\text{localize further}"'] & \mathcal{C}_{A^!} \arrow[l, shift left, "G"] \arrow[d] \\
D(A) & D(A^!)
\end{tikzcd}$$

3) b) Keller (after Lefèvre)

$\boxed{\text{very general!}}$

Idea: ·) look for equivalence

$$D(A) \xrightarrow{\simeq} ?(A^!)$$

·) instead of $A^!$, use dual coalgebra $A^¡ := (A^!)^*$

·) define "coderived category" $coD(A^¡)$ by localizing $K(A^¡)$ at a subset of quasi-isos.

Setup: (1) A dg algebra, $A = \bigoplus_{p \in \mathbb{Z}} A^p$ (only homological grading)

with multiplication μ

augmentation $\varepsilon : A \to k$ $\quad \bar{A} := \ker(\varepsilon)$

(2) C (think $A^¡$) dg coalgebra with

comultiplication Δ

coaugmentation $\eta : k \to C$ $\quad \bar{C} := \operatorname{coker}(\eta)$

– assume that C is cocomplete, i.e. $\bar{C} = \bigcup_{n \geq 2} \ker(\bar{C} \xrightarrow{\Delta} \bar{C}^{\otimes n})$

$\Longrightarrow C^*$ is complete local.

(3) Relate A and C by twisting cochain $\tau : C \to A$ which is

– k-linear

– degree 1

– (i) $\varepsilon \circ \tau \circ \eta = 0$

(ii) $d_A \tau + \tau d_C + \mu(\tau \otimes \tau)\Delta = 0$ (MC condition)

Example: $A = \operatorname{Sym} V$ (V not necessarily finite dim'l)

$C = \operatorname{Sym} V[1]$ $\qquad \tau : C \longrightarrow A$ has one non-zero component identifying the copies of V.

$\rightsquigarrow$ Have adjoints

$$\text{A-Mod} \underset{G = A \otimes_\tau -}{\overset{F = C \otimes_\tau -}{\rightleftarrows}} \overline{\text{C-coMod}}$$

"cocomplete" C-comodules

$FM = C \otimes_k M$ with differential twisted by τ:

$$d_{C \otimes_\tau M} = 1_C \otimes d_M + d_C \otimes 1_M + (1_C \otimes \mu)(1 \otimes \tau \otimes 1)(\Delta \otimes 1_M)$$

Example: $\mathrm{Sym}\, V \otimes_\tau \mathrm{Sym}\, V[1]$ = Koszul ~~[illegible]~~ complex ~~[illegible]~~

Have: $\overline{\text{C-coMod}}$ has notion of weak equivalence stronger than quasi-iso. ("cobar is quasi-iso.")

$\rightsquigarrow \mathrm{coD}(C)$

Prop $D(A) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{coD}(C)$ is an adjunction.

Thm $D(A) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{coD}(C)$ $\sim$ iff $\underbrace{A \otimes_\tau C \otimes_\tau A}_{G(F(A))} \xrightarrow{\sim} A$

cf Koszul bimodule resolution

Example $D(\mathrm{Sym}\, V) \xrightarrow{\sim} \mathrm{coD}(\Lambda V)$

Rem: $F(S_a)$ in ex above $\neq 0$ here!

Remark: Can apply this to $\mathcal{U}\mathfrak{g}$ and $C_*^{Lie}(\mathfrak{g})$

$\Rightarrow D(\mathcal{U}\mathfrak{g}) \xrightarrow{\sim} \mathrm{coD}(C_*^{Lie}(\mathfrak{g}))$